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On the invariance of generating functions for symplectic transformations

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Abstract: This paper treats the invariance of generating functions for symplectic transformations under canonical coordinates transformations. A necessary and sufficient condition for the invariance is obtained. A result of Weinstein [9] is recovered as a special case.

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0. Introduction

Generating functions have been used in classical mechanics for a long time. For example, Poincare used generating functions in his study of celestial mechanics. One type of generating functions which he used is

$$z \longrightarrow \bar{z}: \quad \bar{z} - z = \phi_z\Big(\frac{z+\bar{z}}{2}\Big),$$

where the function ϕ gives rise to a symplectic transformation $z \to \bar{z}$ in $\mathbb{R}^{2n}$. This type of generating functions has become known as Poincare's generating functions. Another type of generating functions often used are the so called generating functions of the first type. In a canonical coordinates (p_i, q_i) they have the form

$$(p_i, q_i) \longrightarrow (\bar{p}_i, \bar{q}_i): \quad p_i = \frac{\partial \phi}{\partial q_i}, \quad \bar{p}_i = -\frac{\partial \phi}{\partial \bar{q}_i},$$

where ϕ is a function on $\mathbb{R}^{2n}$. More recently, Feng et al. [4] have developed a symplectic numerical method for Hamiltonian systems, in which the generating functions are used

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in an essential way (see also [5,6,7,8]). See Arnold [1, Appendix 9], for a discussion about the use of generating functions.

Despite their importance and frequent use, generating functions are not invariantly defined: upon a change of canonical coordinates, the generating function for the same symplectic transformation will take a different form. Nevertheless, Weinstein [9] proved that for the Poincare's generating function, after a symplectic change of coordinates, the new generating functions look the same in *third* coordinates unrelated to the first two. It should be pointed out that here the whole question is that of the critical points of the generating function. This suggests that we may say that the generating functions of a given type are invariant under a symplectic change of coordinates if the new generating functions look the same in *yet another* coordinates (see Section 2 for the definition).

In this paper we study the invariance of generating functions of general type, where the invariance is in the above sense. We obtain a necessary and sufficient condition for the invariance of the generating functions of a given type under a given symplectic transformation. Our method is different from that of Weinstein [9] in that we only use symplectic geometry, while he uses the singularity theory. As a by-product, we show that generating functions invariant under a sufficiently large class of symplectic transformations must be infinitesimally equivalent to Poincare's generating function.

Since the problem is local, we may restrict ourselves to the case of symplectic manifold $T^*\mathbb{R}^n$. The construction of generating functions is based on a symplectic map (cf. Feng [4]) $\alpha : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*(\mathbb{R}^n \times \mathbb{R}^n)$. Modulo some non-transversality conditions, a generating function ϕ_P for a symplectic transformation $P : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ is given by the relation

$$\alpha(x, P(x)) = (w, d\phi_P(w)),$$

where $x \in T^*\mathbb{R}^n$, $w \in \mathbb{R}^{2n}$ (cf. Sect. 1, 2, 3 for details). Let $S : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ be a symplectic transformation (i.e. a canonical transformation of coordinates). Roughly speaking, we say that the generating is invariant under S if for every transformation P satisfying certain transversality conditions,

$$\phi_{S \circ P \circ S^{-1}} = \phi_S \circ A_P + c$$

for a local diffeomorphism $A = A_P$ which depends on P, where c is a constant.

To state our main result, introduce $D(S) : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, $D(S)(z, \bar{z}) = (S(z), S(\bar{z}))$, and $T_S = \alpha \circ D(S) \circ \alpha^{-1}$.

Theorem 1. *The generating functions constructed via α are invariant under S if and only if*

(i) $T_S(Z) = Z$, *where Z is the zero section of $T^*\mathbb{R}^{2n}$;*

(ii) *At any $z \in Z$, the tangent map of T_S can be written as*

$$(T_S)_*(\delta w, \delta\bar{w}) = (B_z \delta w, B_z^T \delta\bar{w}),$$

where $(\delta w, \delta\bar{w}) \in T_z(T^*\mathbb{R}^{2n})$, and B_z is a linear isomorphism of $\mathbb{R}^{2n}$ (depending on z).

If we choose α as follows

$$\alpha_P = \begin{pmatrix} J_n & -J_n \\ I_{2n}/2 & I_{2n}/2 \end{pmatrix}, \quad \text{where} \quad J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

and I_n is the $n \times n$ identity matrix, then we obtain the Poincare's generating function. As a corollary of Theorem 1 (where all the conditions are trivially satisfied), we recover the result of A. Weinstein [9]:

Corollary. *If P, S fix $(0,0) \in T^*\mathbb{R}^n$ and P is given by a generating function ϕ_P, then there is a local diffeomorphism A (depending on P) such that in a neighbourhood of $0 \in \mathbb{R}^{2n}$,*

$$\phi_{S^{-1}\circ P\circ S} = \phi_P \circ A + c,$$

where c is a constant.

1. Basic facts about generating functions

A symplectic manifold is a differentiable manifold together with a closed non-degenerate two-form ω (symplectic two-form). Since the problem we consider in this paper is local, we may assume that the symplectic manifold is $T^*\mathbb{R}^n$. The related symplectic manifolds we will use later on are

1. $T^*\mathbb{R}^n\{z\}$, with the symplectic two-form $\omega = \sum_{i=1}^n dp_i \wedge dq_i$, $z = (p,q)$, $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n)$.
2. $T^*\mathbb{R}^n \times T^*\mathbb{R}^n\{z, \bar{z}\}$, with the symplectic two form $\omega = \sum_{i=1}^n dp_i \wedge dq_i - \sum_{i=1}^n d\bar{p}_i \wedge d\bar{q}_i$, $z = (p,q), \bar{z} = (\bar{p}, \bar{q})$.
3. $T^*\mathbb{R}^{2n}\{w, \bar{w}\}$, with the symplectic two-form $\omega = \sum_{i=1}^{2n} d\bar{w}_i \wedge dw_i$.

A *Lagrangian submanifold* is a submanifold of maximal dimension on which the symplectic two-form vanishes. An important Lagrangian submanifold in $T^*\mathbb{R}^{2n}$ is the *zero section* $Z = \{(w,0), w \in \mathbb{R}^{2n}\} \subset T^*\mathbb{R}^{2n}$.

A symplectic diffeomorphism P of the symplectic manifold is a diffeomorphism which preserves the symplectic two-form ω, i.e. $P^*\omega = \omega$. The graph of a symplectic transformation $P : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$, gra$(P) = \{(z, P(z)), z \in T^*\mathbb{R}^n\}$, is a Lagrangian submanifold in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ (cf. [9]). In particular, the graph of the identity transformation of $T^*\mathbb{R}^n$ is the *diagonal* of $T^*\mathbb{R}^n \times T^*\mathbb{R}^n : \triangle := \{(z,z), z \in T^*\mathbb{R}^n\}$. Conversely, a Lagrangian submanifold in $T^*\mathbb{R}^n \times \mathbb{R}^n$ is the graph of a local symplectic transformation of $T^*\mathbb{R}^n$ iff it intersects with every submanifold $\{z_0\} \times T^*\mathbb{R}^n = \{(z_0, z), z \in T^*\mathbb{R}^n\}$ transversally.

Let $\alpha : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*\mathbb{R}^{2n}$ be a symplectic diffeomorphism. Note that α maps the graph of P, gra(P), to a submanifold L_p, which is a Lagrangian submanifold in $T^*\mathbb{R}^{2n}$. If L_p satisfies the transversality condition:

$$L_p \text{ intersects with the fibers of } T^*\mathbb{R}^{2n} \text{ transversally,} \tag{1}$$

then L_p can be written as

$$L_P = \{(w, d\phi_P(w)), w \in \mathbb{R}^{2n}\}, \tag{2}$$

where ϕ_P is a (locally defined) function on $\mathbb{R}^{2n}$. The function ϕ_P is called the generating function for P constructed via α.

Conversely, a function ϕ on $\mathbb{R}^{2n}$ gives a Lagrangian submanifold $L = \{(w, d\phi(w)), W \in \mathbb{R}^{2n}\}$, and hence a Lagrangian submanifold in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$. This last submanifold is the graph of a symplectic transformation if and only if it

$$\text{intersects every submanifold } \{z_0\} \times T^*\mathbb{R}^n = \{(z_0, z), z \in T^*\mathbb{R}^n\} \text{ transversally} \tag{3}$$

So, once α is given, symplectic transformations and generating functions determine each other, provided that the transversality conditions (1), (3) are satisfied respectively.

2. Definition of the invariance of generating functions

Since the structure of a generating function ϕ_P at a regular point (i.e. $d\phi_P \neq 0$) is trivial, therefore we shall focus our attention on its critical points.

Note that if w is a critical point of ϕ_P, then $(w, 0) \in L_p \cap Z$ and vice versa.

Definition. We say that a generating function constructed via α is *invariant* under a symplectic transformation S if

(i) L_P satisfies the conditions (1), (3) at $L_P \cap Z$ iff $T_S(L_0)$ does so.

(ii) If ϕ_P, $\phi_{S \circ P \circ S^{-1}}$ respectively are the generating functions of P, $S \circ P \circ S^{-1}$, there is a local diffeomorphism $A = A_P$ such that

$$\phi_{S \circ P \circ S^{-1}} = \phi_P \circ A_P + c,$$

where c is a constant.

3. The Necessary Condition

In this section we will prove the necessary condition part of Theorem 1. It follows immediately from the definition that:

Lemma 2. *If the generating function constructed via α is invariant under S, then $T_S(Z) = Z$.*

Proof. We need only to prove that $T_S(Z) \subset Z$. Suppose $W_0 = (w_0, 0) \subset Z$ is such that $T_S(W_0)$ is not in Z. We choose a Lagrangian submanifold L_0 through W_0 which satisfies (1), (3) at W_0. Then the generating function for L_0 has w_0 as a critical point. However, the generating function ϕ_1 for $L_1 := T_S(L_0)$ does not have critical points in a neighbourhood of w_1, where $(w_1, \bar{w}_1) = T_S(w_0, 0)$. So it is impossible for $\phi_1 = \phi_0 \circ A_P + c$ to hold, which is a contradiction. □

We will need some facts from symplectic linear algebra.

Lemma 3. *Suppose that $V_0, V_1, \dots, V_k$ are n-dimensional subspaces in $\mathbb{R}^{2n}$ $(= T^*\mathbb{R}^n$, $V_i \neq V_0$, for $i > 0$. Then there is a Lagrangian subspace $V \subset \mathbb{R}^{2n}$ such that*

$$R^{2n} = V \oplus V_1 = \dots = V \oplus V_k, \;\; V \cap V_0 \neq \{0\}.$$

Proof. We shall make use of the following fact: if $A_1, \dots, A_m$ are subspaces of $\mathbb{R}^n$ of dimension less than n, then $\mathbb{R}^n \neq A_1 \cup A_2 \cup \dots \cup A_m$.

We first choose a vector $x_1 \in V_0$ such that x_1 is not in $\bigcup_{i>0} V_i$. Let B_1 denote the subspace spanned by x_1 and $B_1^{\perp} := \{x, w(x, x_1) = 0\}$. Choose $x_2 \in B_1^{\perp} \cap B_1$ such that x_2 is not in $\bigcup_{i>0} V_i$. Let B_2 be the subspace spanned by x_1, x_2, and $B_2^{\perp}$ the subspace symplectic orthogonal to B_2. We choose $x_3 \in B_2 \cap B_2^{\perp}$ such that x_2 is not in $\bigcup_{i>0} V_i$. Repeat this process, we choose a Lagrangian subspace V which satisfies the condition. □

We will rewrite the transversality conditions (1), (3) in an infinitesimal form. We will write $(w, \bar{w})$ as $\binom{\bar{w}}{w}$. Let K denote the set of Lagrangian subspaces $V \subset T^*\mathbb{R}^{2n}$ satisfying the condition $V \oplus F = \mathbb{R}^{4n}$, and $V \oplus \alpha_*(G) = \mathbb{R}^{4n}$ where $F = \{(0, \bar{w}), \bar{w} \in \mathbb{R}^{2n}\}$, and G is the subspace $G = \{(z, 0), z \in T^*\mathbb{R}^n\}$.

Lemma 4. *Suppose that a linear symplectic transformation $T : T^*\mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$ satisfies the following conditions,*

(i) $T(Z) = Z$;

(ii) $T(V) \in K$ *whenever* $V \in K$;

then

$$T = \begin{pmatrix} B & 0 \\ 0 & B^{-T} \end{pmatrix},$$

where B is a non-singular matrix.

Proof. Assume

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

Then (i) implies $T_{12} = 0, T_{22} = T_{11}^{-1}$. Set

$$T_1 := T \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & T_{11}^{\mathsf{T}} \end{pmatrix} := \begin{pmatrix} I_{2n} & 0 \\ S & I_{2n} \end{pmatrix}.$$

Then this new transformation still satisfies the conditions (i) and (ii).

If $S \neq 0$, then $F \neq T_1(F)$. Applying Lemma 3, we can choose a Lagrangian subspace $V \in K$ such that $V \cap T_1^{-1}(F) \neq \{0\}$ and hence $T_1(V) + F \neq \mathbb{R}^{4n}$. This means $T_1(V)$ is not in K while $V \in K$, which is a contradiction. So $S = 0$. □

We now prove the necessary condition part of Theorem 1. (i) has already been proved in Lemma 2, we need only to prove (ii).

Now the tangent map $(T_S)_*$ on Z has the property that a Lagrangian subspace L in the tangent space to $T^*\mathbb{R}^{2n}$ satisfies (1), (3) iff $(T_S)_*(L)$ does. Applying Lemma 4, we prove (ii). $\square$

4. The Sufficient Condition

In the section, we shall prove the sufficient condition part of Theorem 1. We assume that $T_S : T^*\mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$ satisfies (i), (ii) in Theorem 1.

Lemma 5. *The map T_S can be decomposed into*

$$T_S = T_1 \circ T_2$$

where

(i) *T_2 is a symplectic transformation of the form*

$$T_2(w, \bar{w}) = (B(w), B_w^{-*}\bar{w}),$$

where B is a diffeomorphism of $\mathbb{R}^{2n}$.

(ii) *$T_1(W) = W$, for any $W \in Z$, and at Z, the tangent map of T_1 is the identity transformation.*

Proof. Since $T_S(Z) = Z$, by restriction to Z, T_S induces a diffeomorphism B of Z. Set

$$T_2(w, \bar{w}) = (B(w), B_w^{-*}\bar{w}).$$

Then $T_1 = T_S \circ T_2^{-1}$ satisfies the condition (ii). $\square$

Now we need only to prove the theorem for T_1. We shall construct a homotopy of symplectic transformations joining Id to T_1.

Lemma 6. *There is a one-parameter family of symplectic transformations T^t ($0 \leqslant t \leqslant 1$) defined on a neighbourhood of $Z \in T^*\mathbb{R}^{2n}$, such that $T^0 = I$, $T^1 = T_1$, $T^t(W) = W$ for all $W \in Z$, and the tangent map of T^t at $W \in Z$ is the identity map.*

Proof. We shall use Poincare's generating function to construct T^t.

Let φ_t be the generating function for T^t,

$$W_t = T^t(W), \qquad W_t - W = J^{-1}\varphi_w^t\Big(\frac{W + W_t}{2}\Big). \tag{4}$$

Since the tangent maps of T_1 at Z are the identity transformations, in a neighbourhood of Z, T_1 can be given by a generating function φ^1.

Now α_P (for the Poincare's generating function) maps the submanifold $U := \{(W, W), W \in Z\} \subset T^*\mathbb{R}^{2n} \times T^*\mathbb{R}^{2n}$ to the following linear subspace in the zero-section of $T^*\mathbb{R}^{4n}$

$$V = \{(Y, 0);\ Y = (y, 0),\ y \in \mathbb{R}^{2n}\}.$$

Symplectic transformations T^t satisfies our conditions iff its generating function $\phi^t(W)$ together with its first and second derivatives vanish on the submanifold $V_0 = \{(y,0), y \in \mathbb{R}^{2n}\} \subset \mathbb{R}^{4n}$.

Note that the generating function for the identity transformation is $\varphi^0 = 0$. Set $\varphi^t = t\varphi^1$, then the corresponding symplectic transformations are what we ask for. □

We may assume that T^t is the phase flow of a time-dependent Hamiltonian system with Hamiltonian $H(\cdot,t)$, which is normalized to $H(0,t) = 0$.

Let $L_0 \subset T^*\mathbb{R}^{2n}$ be a Lagrangian submanifold which satisfies the transversality conditions (1), (3). Then the one-parameter family of Lagrangian submanifolds $L_t := T^t(L_0)$ also have this property. Let ϕ^t be the generating functions of L_t, then a classical result in the theory Hamilton–Jacobi equation says that the function $\phi(w,t) = \phi^t(w)$ satisfies

$$\frac{\partial\phi(w,t)}{\partial t} = H(w, d\phi^t(w), t). \tag{5}$$

This result follows from the following facts. First recall that the submanifold $N = \{(w, \bar{w}, t, H(w,\bar{w},t)), (w,\bar{w}) \in L_t, t \in \mathbb{R}\}$ is a Lagrangian submanifold in $T^*(\mathbb{R}^{2n} \times \mathbb{R})$ with respect to the symplectic two-form $\sum d\bar{w}_i \wedge dw_i - dt \wedge dE$. So there is a function $\varphi(w,t)$, with $\varphi(0,0) = 0$, such that locally

$$N = \left\{\left(w, \frac{\partial\varphi}{\partial w}, t, \frac{\partial\varphi}{\partial t}\right), (w,t) \in \mathbb{R}^{2n} \times \mathbb{R}\right\}.$$

So

$$\frac{\partial\varphi(w,t)}{\partial w} = \frac{\partial\phi^t(w)}{\partial w}, \quad \frac{\partial\varphi(w,t)}{\partial t} = H\left(w, \frac{\partial\varphi(w,t)}{\partial w}, t\right).$$

From the later relation we have

$$\frac{\partial\varphi(0,t)}{\partial t} = 0,$$

and hence $\varphi(0,t) = 0$. So $\varphi(w,t) = \phi^t(w)$ and hence $\phi^t(w)$ satisfies the above Hamilton–Jacobi equation.

We are now ready to prove the sufficient condition in Theorem 1. As mentioned, we need only to prove the theorem for T_1.

Let T^t be the one-parameter family of symplectic transformations as in Lemma 6, which is the phase flow of a time-dependent Hamiltonian system $H(W,t)$. Then $H(W,t)$ vanishes on the zero section Z. By Malgrange preparation theorem [2], there are smooth functions $q_1, \ldots, q_{2n}$ defined in a neighbourhood of Z, such that

$$H(W,t) = \bar{w}_1 q_1(W,t) + \ldots + \bar{w}_{2n} q_{2n}(W,t),$$

where $W = (w, \bar{w}), \bar{w} = (\bar{w}_1, \ldots, \bar{w}_{2n})$.

Suppose $L_t = T^t(L_0)$ is the one-parameter family of Lagrangian submanifolds as above, then from (5) we obtain (cf. [1])

$$\frac{\partial\phi(w,t)}{\partial t} = \frac{\partial\phi(w,t)}{\partial w_1} r_1(w,t) + \ldots + \frac{\partial\phi(w,t)}{\partial w_{2n}} r_{2n}(w,t) \tag{6}$$

where $r_i(w,t) = q_i(w, d\phi^t(w), t)$, $i = 1, \ldots, 2n$. Hence, let A^t be the phase flow on $\mathbb{R}^{2n}$ generated by

$$\frac{dw_1}{dt} = r_1(w,t), \quad \ldots, \quad \frac{dw_{2n}}{dt} = r_{2n}(w,t)$$

then from (6), we obtain $\phi^t(w) = \phi^0 \circ A^t(w)$. In particular, $\phi^1 = \phi^0 \circ A^1$. □

5. Generating functions invariant under any symplectic transformation

Lemma 7. *If a generating function constructed via α is invariant under any symplectic transformation and there is a point $(z_0, z_0) \in \Delta$ such that $\alpha(z_0, z_0) \in Z$, then $\alpha(\Delta) = Z$.*

Proof. Choose S to be the phase flow of the Hamiltonian system $f_i = z_i^3$, $S = G^t_{f_i}$, then $T_S = T_{G^t_{f_i}}$ is the phase flow of $\bar{F}_i = F_i \circ \alpha^{-1}$, where $F_i(z, \bar{z}) = f_i(z) - f_i(\bar{z})$. By Lemma 2, $T_{G^t_{f_i}}(Z) = Z$, hence the restriction of $\bar{F}_i$ to Z is a constant c. On the other hand, $\bar{F}_i(\alpha(z_0, z_0)) = f_i(z_0) - f_i(z_0) = 0$ and hence $c = 0$. So we have $Z \subset \bar{F}_i^{-1}(0)$, or $\alpha^{-1}(Z) \subset F_i^{-1}(0) = \{(z, \bar{z}), z = (z_1, \ldots, z_{2n}), \bar{z} = (\bar{z}_1, \ldots, \bar{z}_{2n}), z_i = \bar{z}_i\}$, and hence $\alpha^{-1}(Z) \subset \bigcap_{i=1}^{2n} F_i^{-1}(0) = \Delta$. Similarly, we can prove that $\alpha(\Delta) \subset Z$. So $\alpha^{-1}(Z) = \Delta$. □

Remark. The property that $\alpha(\Delta) = Z$ is significant in that if a symplectic transformation P can be given by a generating function ϕ_P, then there is a one-to-one correspondence between the critical points of ϕ_P and the fixed points of P ([1, Appendix 9], [3]).

We need the following facts of symplectic linear algebra:

Lemma. *If $C \in GL(2n)$ commutes with any $S \in \mathrm{Sp}(n)$ (the group of linear symplectic transformations of $\mathbb{R}^{2n}$ $(= T^*\mathbb{R}^n)$) then $C = \lambda_0 I_{2n}$.*

The proof of this lemma is somehow standard, which we omit.

Lemma 8. *Suppose $\beta \in GL(4n)$ satisfies*

$$\beta^{\mathsf{T}} J_{4n} \beta = \bar{J}_{4n}, \quad J_{4n} = \begin{pmatrix} 0 & I_{2n} \\ -I_{2n} & 0 \end{pmatrix}, \quad \bar{J}_{4n} = \begin{pmatrix} -J_{2n} & 0 \\ 0 & J_{2n} \end{pmatrix} \tag{7}$$

and for any $s \in \mathrm{Sp}(n)$ there is $v \in GL(2n)$ such that

$$\beta \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \beta^{-1} = \begin{pmatrix} v & 0 \\ 0 & v^{-T} \end{pmatrix} \tag{8}$$

then β can be written either as

$$\beta = \begin{pmatrix} A & -A \\ D & D \end{pmatrix}, \quad \text{where} \quad A^{\mathsf{T}} D = -D^{\mathsf{T}} A = -J_{2n}/2,$$

or as

$$\beta = \begin{pmatrix} A & A \\ D & -D \end{pmatrix}, \quad \textit{where} \quad A^{\mathsf{T}}D = -D^{\mathsf{T}}A = -J_{2n}/2.$$

Proof. Let

$$\beta = \begin{pmatrix} A_\beta & B_\beta \\ C_\beta & -D_\beta \end{pmatrix}.$$

From (8) we obtain

$$A_\beta s = vA_\beta, \quad B_\beta s = vB_\beta, \quad S_\beta s = v^{-T}C_\beta, \quad D_\beta s = v^{-T}D_\beta. \tag{9}$$

We want to prove that there are numbers $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$ and matrices A, D such that

$$A_\beta = \lambda_{11}A, \quad B_\beta = \lambda_{12}A, \quad C_\beta = \lambda_{21}D, \quad D_\beta = \lambda_{22}D. \tag{10}$$

(i) If $\det(A_\beta) \neq 0$, then it follows from (9) that $(A_\beta^{-1}B_\beta)s = s(A_\beta^{-1}B_\beta)$, hence $A_\beta^{-1}B_\beta = \lambda I$.

(ii) If $\det(A_\beta) = 0$, then $\operatorname{Ker} A_\beta$ is not empty. Then it follows from (9) that if $x \in \operatorname{Ker} A_\beta$, then $s(x) \in \operatorname{Ker} A_\beta$. Hence $\operatorname{Ker} A_\beta$ is invariant under any $s \in \mathrm{Sp}(n)$. On the hand, for any $a, b \in \mathbb{R}^{2n} - \{0\}$, there is an $s \in \mathrm{Sp}(n)$ such that $s(a) = b$. So $\operatorname{Ker} A_\beta = \mathbb{R}^{2n}$, or $A_\beta = 0$.

In both cases, we proved (10) .

Now since β must satisfy (7), we have

$$\lambda_{11}\lambda_{21}(A^{\mathsf{T}}D - D^{\mathsf{T}}A) = -J_{2n}, \quad \lambda_{12}\lambda_{22}(A^{\mathsf{T}}D - D^{\mathsf{T}A}) = J_{2n}, \tag{11}$$

$$\lambda_{12}\lambda_{21}A^{\mathsf{T}}D - \lambda_{22}\lambda_{11}D^{\mathsf{T}}A = 0, \quad \lambda_{11}\lambda_{22}A^{\mathsf{T}}D - \lambda_{21}\lambda_{12}D^{\mathsf{T}}A = 0. \tag{12}$$

We may assume, without loss of generality, that $\lambda_{11} = 1$, $\lambda_{12} = \lambda_1$, $\lambda_{21} = \lambda_2$, $\lambda_{22} = 1$. From (19), we obtain $\lambda_1^2 = 1$, and the lemma follows. □

Theorem 9. *Suppose that* $\alpha : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*\mathbb{R}^{2n}$ *is a symplectic diffeomorphism with the property* $\alpha(\Delta) = Z$, $\alpha(0,0) = (0,0)$. *Then* α *has the following property that whenever symplectic transformations* P, S *fixing* $(0,0) \in T^*\mathbb{R}^n$,

$$\phi_{S \circ P \circ S^{-1}} = \phi_P \circ A_P + c,$$

where A_P *is a local diffeomorphism, if and only if at* $(0,0) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, *the tangent map of* α *has the form*

$$\alpha_*(\delta z, \delta \bar{z}) = (\tfrac{1}{2}B(\delta z + \delta \bar{z}),\ \tfrac{1}{2}A(\delta \bar{z} - \delta z)).$$

Proof. This follows from Theorem 1 and Lemma 8. □

6. Special cases

Consider a linear symplectic map

$$\alpha : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*\mathbb{R}^{2n}, \quad \alpha(z, \bar{z}) = (C_\alpha \bar{z} + D_\alpha z, A_\alpha \bar{z} + B_\alpha z),$$

which can be represented as a matrix

$$\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ C_\alpha & D_\alpha \end{pmatrix}.$$

Suppose $\det(C_\alpha + D_\alpha) \neq 0$, Feng Kang (unpublished) has shown that the generating function constructed via α can be reduced to that constructed via

$$\alpha_0 = \begin{pmatrix} J & -J \\ (I+b)/2 & (I-b)/2 \end{pmatrix},$$

where b satisfies $b^\mathsf{T} J + Jb = 0$. Hence, we may only consider those generating functions constructed via the above α_0.

In this case, Theorem 1 gives

Corollary 10. *Suppose S fixes $(0,0) \in T^*\mathbb{R}^n$, and its tangent map at $(0,0)$ satisfies $S_*(0)b = bS_*(0)$, then, for a symplectic transformation which fixes $(0,0)$ and which can be given by a generating function ϕ_P, there is a local diffeomorphism A_P such that in a neighbourhood of $0 \in \mathbb{R}^{2n}$,*

$$\phi_{S \circ P \circ S^{-1}} = \phi_P \circ A_P + c.$$

The result of Weinstein [9] corresponds to the special case $b = 0$.

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